Love-for-Variety

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Introduction

Love-for-Variety (LV): Utility (productivity) gains from increasing variety of consumer goods (inputs).

- A natural consequence of the quasi-concavity of the utility (production) function. That is, the convexity of the indifference curve (isoquant curve).

- Roughly speaking, for a symmetric CRS production function, $X(\mathbf{x})$, Love-for-Variety (LV) may be measured as:

$$\mathcal{L}(V) \equiv \frac{\partial \ln W(V)}{\partial \ln V} > 0,$$

where V is the variety of available inputs and

$$W(V) \equiv \max_{\mathbf{x}} \left\{ X(\mathbf{x}) \middle| \int_0^V x_\omega d\omega \leq 1. \right\}.$$

- Following the work of Dixit-Stiglitz (1977), Krugman (1980), Ethier (1982), Romer (1987), LV has become a central concept in economic growth, international trade, and economic geography.

- Commonly discussed in monopolistic competition settings, but also useful in other contexts, such as gains from trade in Armington-type competitive models of trade.

- The literature mostly discusses LV under the CES assumption.

LV measure under CES:

$$X(\mathbf{x}) = Z\left[\int_{\Omega} x_{\omega}^{1-\frac{1}{\sigma}} d\omega\right]^{\frac{\sigma}{\sigma-1}} \Rightarrow \mathcal{L}(V) = \frac{1}{\sigma - 1} > 0$$

where $Z > 0$ is TFP and $\sigma > 1$ captures *TWO related but distinct* concepts,

- Elasticity of Substitution (ES) across different goods
- Price Elasticity (PE) of demand for each good.

Two Appealing Features:

- LV is inversely related to ES (and PE).
- Knowing PE tells you everything you need to know about ES and LV.
  CES has only one parameter, σ, so anything related to CES is a function of σ only.

Two Unappealing Features

- LV is independent of V, the variety of available goods. Intuitively, LV should decline as V increases.
  In this regard, some may find "ideal-variety approach," or "Lancaster's characteristic theory" more appealing, but they are less tractable & less flexible than "Love-for-Variety approach."
- The relation btw PE, ES, & LV are hard-wired under CES, with no flexibility.
  To "account for" the gap btw the revealed LV and CES-implied LV, one need to introduce "the Benassy residual," or "quality-adjustment," whose estimate depends on CES.

The Questions: How does the LV measure need to be modified if we move away from CES?

- How is LV related to the underlying demand structure, such as ES or PE?

ES and PE are distinct concepts outside of CES, which could play different roles shaping LV.

- How biased are our estimates of LV & of the "Benassy residuals" or "quality-term" if we incorrectly assume CES?
- Under what conditions does LV decline as the variety of available goods increases?

*Does it help to introduce the empirically plausible **Marshall's 2nd Law of demand** (PE higher at a higher price)?*

- Can we develop "Love-for-variety approach" with diminishing LV, which is also tractable?

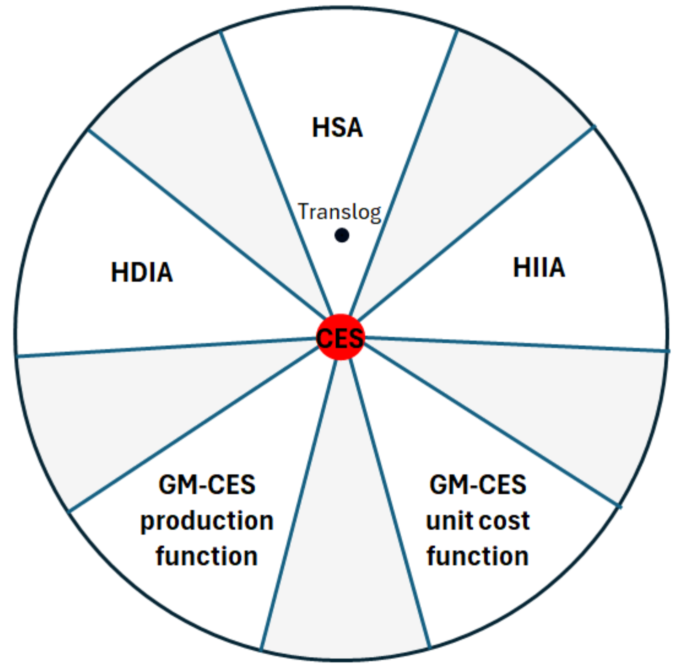
Our Approach:

- Define the two measures: **Substitutability,** $\mathcal{S}(V)$, & **Love-for-Variety,** $\mathcal{L}(V)$, under homotheticity and symmetry
  - Both depend only on V (the variety of available goods).
  - Under CES, $\mathcal{S}(V)$ and $\mathcal{L}(V)$ are both constant and $\mathcal{L}(V) = 1/(\sigma - 1) = 1/(\mathcal{S}(V) - 1)$.
- Are there non-CES under which $\mathcal{S}(V)$ and/or $\mathcal{L}(V)$ are independent of V?
  - The answer turns out to be "yes".
- What if $\mathcal{S}(V)$ varies with V?
  - One might intuitively think "**The 2nd Law of demand** $\Longrightarrow$ **Increasing** $\mathcal{S}(V)$ $\Longrightarrow$ **Diminishing** $\mathcal{L}(V)$."
  - There are classes of demand systems under which this is true, but not true in general.
- The CES formula may
  - underestimate LV; $\mathcal{L}(V) > 1/(\mathcal{S}(V) - 1)$, hence overestimate "Benassy Residuals" or "quality-term".
  - overestimate LV; $\mathcal{L}(V) < 1/(\mathcal{S}(V) - 1)$, hence underestimate "Benassy Residuals" or "quality-term"

Anything goes. Homotheticity & symmetry alone impose little restrictions btw PE, $\mathcal{S}(V)$ & $\mathcal{L}(V)$.

Five Classes of non-CES

pairwise disjoint with the sole exception of CES.



Homothetic symmetric demand systems with gross substitutes

Two Classes: GM-CES

obtained by taking the weighted geometric means of CES unit cost or production functions with heterogenous σ.

Theorem 1: Under GM-CES, $\mathcal{S}(V)$ & $\mathcal{L}(V)$ are both independent of V and

$$\mathcal{L}(V) \equiv \mathcal{L}^{GMCES} > \frac{1}{\mathcal{S}^{GMCES} - 1} \equiv \frac{1}{\mathcal{S}(V) - 1}$$

unless CES.

- $\mathcal{S}(V) \equiv \mathcal{S}^{GMCES}$ determines the lower bound of $\mathcal{L}(V) \equiv \mathcal{L}^{GMCES}$. Unbounded from above.
- The CES formula underestimates $\mathcal{L}(V) \equiv \mathcal{L}^{GMCES}$ and overestimates the Benassy residuals or quality improvement term, potentially by a wide margin.

Three Classes: H.S.A., HDIA, and HIIA.

- PE $= \zeta_\omega \equiv \zeta(p_\omega/\mathcal{A}(\mathbf{p}))$, where $\mathcal{A}(\mathbf{p})$ is linear homogeneous, a sufficient statistic for the cross-price effects.

Theorem 2: Under H.S.A., HDIA, and HIIA,

i) $\zeta'(p_\omega/\mathcal{A}(\mathbf{p})) \gtreqqless 0 \Leftrightarrow \mathcal{S}'(V) \gtreqqless 0.$

ii) $\mathcal{S}'(V) \gtreqqless 0$ for all $V > 0 \Longrightarrow \mathcal{L}'(V) \lesseqqgtr 0$ for all $V > 0$. The converse is not true.

iii) $\mathcal{L}'(V) = 0$ for all $V > 0 \Leftrightarrow \mathcal{S}'(V) = 0$ for all $V > 0$, which occurs iff CES.

Theorem 3: $\mathcal{L}'(V) \lesseqqgtr 0 \Leftrightarrow \mathcal{L}(V) \lesseqqgtr 1/(\mathcal{S}(V) - 1)$.

The 2nd Law
$\zeta(p_\omega; \mathbf{p})$ is increasing in p_ω
$\zeta^*(x_\omega; \mathbf{x})$ is decreasing in x_ω

$\Updownarrow$

Increasing Substitutability
$\mathcal{S}'(V) > 0$ for all $V > 0$.

$\Rightarrow$

Diminishing Love-for-Variety
$\mathcal{L}'(V) < 0$ for all $V > 0$.

$\Leftrightarrow$

The CES formula overestimates Love-for-Variety.

$$\mathcal{L}(V) < \frac{1}{\mathcal{S}(V) - 1}$$

Theorem 4: As $V \to \infty$, $\mathcal{L}(V) - 1/(\mathcal{S}(V) - 1) \to 0$.

An Application: Gains from Trade in an Armington Model of Competitive Trade

- Theorems 1-4 are all about the *demand system*, independent of the supply-side. They hold regardless of how the variety changes. E.g., the variety may change due to international trade, pure discovery, innovation by the public sector, or by the private sector, which could be monopolistic, oligopolistic, or monopolistically competitive, etc.

- Here, we illustrate the implications in a simple Armington model of trade btw 2 countries, which produce different sets of goods competitively and differ in the variety of goods they produce = country size

	Home	**Foreign**
Domestic Expenditure Share	$\lambda = \dfrac{V}{V + V^*}$	$\lambda^* = \dfrac{V^*}{V + V^*}$
Gains from Trade	$\ln(GT) = \displaystyle\int_{V}^{V+V^*} \mathcal{L}(v)\frac{dv}{v}$	$\ln(GT^*) = \displaystyle\int_{V^*}^{V+V^*} \mathcal{L}(v)\frac{dv}{v}$

Among other things, we show:

- Under the 2 classes of GM-CES, $\mathcal{L}(v) = \mathcal{L}^{GMCES}$ and hence,

$$\ln(GT) = \mathcal{L}^{GMCES} \ln\left(\frac{1}{\lambda}\right) > \frac{1}{\mathcal{S}^{GMCES} - 1} \ln\left(\frac{1}{\lambda}\right).$$

  unless CES.

The ACR formula holds with $\mathcal{L}^{GMCES}$, not with $1/[\mathcal{S}^{GMCES} - 1]$. The CES formula underestimates GT, potentially by a wide margin. A possible solution for "elusive gains from trade."

- Under the 3 classes: λ is no longer a sufficient statistic for GT.

  - Controlling for the relative country size, hence for λ, GT changes with the absolute sizes of the two countries.
    GT is larger btw two smaller countries than btw two larger countries under diminishing LV.

  - A smaller λ increases GT, but its implications depend on whether it is due to a smaller V or due to a larger V^*.
    - E.g., With the choke price, GT is increasing in the size of the trading partner, but it is bounded.
      Under CES, it is unbounded. CES may overestimate gains from trade with a large country.

Notes:

- Though some existing studies have looked at $\mathcal{S}(V)$ under some parametric families of homothetic symmetric non-CES, none have looked at $\mathcal{L}(V)$, or they took for granted that $\mathcal{L}(V) = 1/(\mathcal{S}(V) - 1)$ would continue to hold under non-CES.

- Neither symmetry nor homotheticity are as restrictive as they look.
  - By nesting symmetric homothetic demand systems into an upper-tier asymmetric/nonhomothetic demand system, we can create an asymmetric/nonhomothetic demand system.
  - Homotheticity is indeed *an advantage*, which makes non-CES applicable to a sector-level analysis in multi-sector settings.
  - Moreover, one key message is that symmetry/homotheticity are *not restrictive enough--* "Anything goes,"-- that we need to look for more restrictions to make further progress.

Plan of the Talk

- Introduction
- General Symmetric Homothetic Demand Systems
  - Substitutability Measure, $\mathcal{S}(V)$.
  - Love-for-Variety Measure, $\mathcal{L}(V)$.
- Two Classes of Geometric Means of CES.
- Three Classes of H.S.A., HDIA, and HIIA.
- An Application to an Armington Model of Trade
- Concluding Remarks

General Symmetric Homothetic Demand Systems

General Symmetric Homothetic (Input) Demand System

Consider demand system for a continuum of differentiated inputs generated by symmetric CRS production technology.

CRS Production Function	Unit Cost Function
$X(\mathbf{x}) \equiv \min_{\mathbf{p}} \left\{ \mathbf{px} = \int_{\Omega} p_\omega x_\omega d\omega \middle| P(\mathbf{p}) \geq 1 \right\}$	$P(\mathbf{p}) \equiv \min_{\mathbf{x}} \left\{ \mathbf{px} = \int_{\Omega} p_\omega x_\omega d\omega \middle| X(\mathbf{x}) \geq 1 \right\}$

$\mathbf{x} = \{x_\omega; \omega \in \overline{\Omega}\}$**:** the input quantity vector; $\mathbf{p} = \{p_\omega; \omega \in \overline{\Omega}\}$**:** the input price vector.

$\overline{\Omega}$, the continuum set of all potential inputs. $\Omega \subset \overline{\Omega}$, the set of available inputs with its mass $V \equiv |\Omega|$.

$\overline{\Omega} \backslash \Omega$: the set of unavailable inputs, $x_\omega = 0$ and $p_\omega = \infty$ for $\omega \in \overline{\Omega} \backslash \Omega$.

Inputs are *inessential*, i.e., $\overline{\Omega} \backslash \Omega \neq \emptyset$ does NOT imply $X(\mathbf{x}) = 0 \Leftrightarrow P(\mathbf{p}) = \infty$.

Duality: Either $X(\mathbf{x})$ or $P(\mathbf{p})$ can be a *primitive*, if linear homogeneity, monotonicity & strict quasi-concavity satisfied

Demand System

Demand Curve (from Shepherd's Lemma)	Inverse Demand Curve
$x_\omega = \dfrac{\partial P(\mathbf{p})}{\partial p_\omega} X(\mathbf{x})$	$p_\omega = P(\mathbf{p}) \dfrac{\partial X(\mathbf{x})}{\partial x_\omega}$

From Euler's Homogenous Function Theorem,

$$\mathbf{px} = \int_{\Omega} p_\omega x_\omega d\omega = \int_{\Omega} p_\omega \left[\frac{\partial P(\mathbf{p})}{\partial p_\omega} X(\mathbf{x}) \right] d\omega = \int_{\Omega} \left[p_\omega \frac{\partial P(\mathbf{p})}{\partial p_\omega} \right] X(\mathbf{x}) d\omega = P(\mathbf{p}) X(\mathbf{x}) = E.$$

The value of inputs is equal to the total value of output under CRS.

Budget Share of $\omega \in \Omega$:	$s_\omega \equiv \frac{p_\omega x_\omega}{\mathbf{px}} = \frac{p_\omega x_\omega}{P(\mathbf{p})X(\mathbf{x})} = \frac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} \equiv s(p_\omega, \mathbf{p}) = \frac{\partial \ln X(\mathbf{x})}{\partial \ln x_\omega} \equiv s^*(x_\omega, \mathbf{x})$

Homogeneity of degree zero → $s_\omega = s(1, \mathbf{p}/p_\omega) = s^*(1, \mathbf{x}/x_\omega)$.

In general, it depends on the whole *distribution* of the prices (quantities) divided by its own price (quantity).

Definition: Gross Substitutability $$\frac{\partial \ln s(p_\omega; \mathbf{p})}{\partial \ln p_\omega} < 0 \Leftrightarrow \frac{\partial \ln s^*(x_\omega;\ \mathbf{x})}{\partial \ln x_\omega} > 0$$

The choke price $\bar{p}$ exists if $s_\omega = s(1, \mathbf{p}/p_\omega) = 0$ for all $p_\omega \geq \bar{p}$. (The choke price depends on the price vector.)

Under CES, $\sigma > 1$ ensures both inessentiality & gross substitutability. In general, they need to be assumed separately.

Price Elasticity of Demand for $\omega \in \Omega$	$\zeta_\omega \equiv -\frac{\partial \ln x_\omega}{\partial \ln p_\omega} = \zeta(p_\omega; \mathbf{p}) \equiv 1 - \frac{\partial \ln s(p_\omega; \mathbf{p})}{\partial \ln p_\omega} = \zeta^*(x_\omega; \mathbf{x}) \equiv \left[1 - \frac{\partial \ln s^*(x_\omega;\ \mathbf{x})}{\partial \ln x_\omega}\right]^{-1} > 1.$

Homogeneity of degree zero → $\zeta_\omega = \zeta(1, \mathbf{p}/p_\omega) = \zeta^*(1, \mathbf{x}/x_\omega)$.

In general, it depends on the whole *distribution* of prices (quantities) divided by its own price (quantity).

Under CES, it doesn't depend on the prices at all.

Definition: The 2nd Law of Demand $$\frac{\partial \ln \zeta(p_\omega; \mathbf{p})}{\partial \ln p_\omega} > 0 \Leftrightarrow \frac{\partial \ln \zeta^*(x_\omega; \mathbf{x})}{\partial \ln x_\omega} < 0.$$

Clearly, CES does not satisfy the 2nd Law.

Substitutability Measure Across Different Goods

Unit Quantity Vector: $\mathbf{1}_\Omega \equiv \{(1_\Omega)_\omega; \omega \in \bar{\Omega}\}$, where $(1_\Omega)_\omega \equiv \begin{cases} 1 & \text{for} \quad \omega \in \Omega \\ 0 & \text{for} \quad \omega \in \bar{\Omega}\backslash\Omega \end{cases}$

Unit Price Vector: $\mathbf{1}_\Omega^{-1} \equiv \left\{\left(1_\Omega^{-1}\right)_\omega; \omega \in \bar{\Omega}\right\}$, where $\left(1_\Omega^{-1}\right)_\omega \equiv \begin{cases} 1 & \text{for} \quad \omega \in \Omega \\ \infty & \text{for} \quad \omega \in \bar{\Omega}\backslash\Omega \end{cases}$

Note: $\int_\Omega (1_\Omega)_\omega d\omega = \int_\Omega \left(1_\Omega^{-1}\right)_\omega d\omega = |\Omega| \equiv V$.

At the symmetric patterns, $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$ and $\mathbf{x} = x\mathbf{1}_\Omega$,

$$s_\omega = s(1, \mathbf{p}/p_\omega) = s^*(1, \mathbf{x}/x_\omega) = s\left(1, \mathbf{1}_\Omega^{-1}\right) = s^*(1, \mathbf{1}_\Omega) = 1/V$$

$$\zeta_\omega = \zeta(1, \mathbf{p}/p_\omega) = \zeta^*(1, \mathbf{x}/x_\omega) = \zeta\left(1, \mathbf{1}_\Omega^{-1}\right) = \zeta^*(1, \mathbf{1}_\Omega) > 1$$

Clearly, this depends only on V. We propose:

Definition: *The substitutability measure across goods* is defined by

$$\mathcal{S}(V) \equiv \zeta\left(1; \mathbf{1}_\Omega^{-1}\right) = \zeta^*(1; \mathbf{1}_\Omega) > 1.$$

We call the case of $\mathcal{S}'(V) > (<)0$ for all $V > 0$, the case of ***increasing (decreasing) substitutability***.

Notes:

- We also consider the alternative definition of $\mathcal{S}(V)$ in terms of Allen-Uzawa Elasticity of Substitution evaluated at the symmetric patterns. Perhaps surprisingly, it turns out to be equivalent.
- In general, the 2^{nd} Law is neither sufficient nor necessary for increasing substitutability, $\mathcal{S}'(V) > 0$.

Love-for-Variety Measure: Commonly defined as the rate of productivity gain from a higher V, holding xV

$$\mathcal{L}(V) \equiv \frac{\partial \ln W(V)}{\partial \ln V} = \frac{d \ln X(\mathbf{x})}{d \ln V}\bigg|_{\mathbf{x}=x\mathbf{1}_{\Omega}, xV=const.} = \frac{d \ln xX(\mathbf{1}_{\Omega})}{d \ln V}\bigg|_{xV=const.} = \frac{d \ln X(\mathbf{1}_{\Omega})}{d \ln V} - 1 > 0$$

Alternatively, LV may be defined as the rate of decline in $P(\mathbf{p})$ from a higher V, at $\mathbf{p} = p\mathbf{1}_{\Omega}^{-1}$, holding p constant.

$$-\frac{d \ln P(\mathbf{p})}{d \ln V}\bigg|_{\mathbf{p}=p\mathbf{1}_{\Omega}^{-1},\ p=const.} = -\frac{d \ln P\left(\mathbf{1}_{\Omega}^{-1}\right)}{d \ln V} > 0.$$

Both are functions of V only, and equivalent because, by applying $\mathbf{x} = x\mathbf{1}_{\Omega}$ and $\mathbf{p} = p\mathbf{1}_{\Omega}^{-1}$ to $\mathbf{px} = P(\mathbf{p})X(\mathbf{x})$,

$$pxV = pP\left(\mathbf{1}_{\Omega}^{-1}\right)xX(\mathbf{1}_{\Omega}) \Longrightarrow -\frac{d \ln P\left(\mathbf{1}_{\Omega}^{-1}\right)}{d \ln V} = \frac{d \ln X(\mathbf{1}_{\Omega})}{d \ln V} - 1 > 0.$$

Definition. *The love-for-variety measure* is defined by:

$$\mathcal{L}(V) \equiv -\frac{d \ln P\left(\mathbf{1}_{\Omega}^{-1}\right)}{d \ln V} = \frac{d \ln X(\mathbf{1}_{\Omega})}{d \ln V} - 1 > 0.$$

We call the case of $\mathcal{L}'(V) < (>) 0$ for all $V > 0$, the case of ***diminishing (increasing) love-for-variety***.

Note: $\mathcal{L}(V) > 0$ is guaranteed by the strict quasi-concavity.

Standard CES with Gross Substitutes:

$$X(\mathbf{x}) = Z\left[\int_{\Omega} x_{\omega}{}^{1-\frac{1}{\sigma}} d\omega\right]^{\frac{\sigma}{\sigma-1}} \Leftrightarrow P(\mathbf{p}) = \frac{1}{Z}\left[\int_{\Omega} p_{\omega}{}^{1-\sigma} d\omega\right]^{\frac{1}{1-\sigma}},$$

where $\sigma > 1$. ($Z > 0$ is TFP or affinity in the preference, in the context of spatial economics)

	CES
Budget Share	$s_{\omega} = \left(\frac{p_{\omega}}{ZP(\mathbf{p})}\right)^{1-\sigma} = \left(\frac{Zx_{\omega}}{X(\mathbf{x})}\right)^{1-1/\sigma}$
Price Elasticity	$\zeta_{\omega} = \sigma > 1$
Substitutability	$\mathcal{S}(V) = \sigma > 1$
Love-for-variety	$\mathcal{L}(V) = \frac{1}{\sigma - 1} > 0.$

Under Standard CES,

- PE, $\zeta(p_{\omega}; \mathbf{p}) = \zeta^{*}(x_{\omega}; \mathbf{x})$, is independent of $\mathbf{p}$ or $\mathbf{x}$ and equal to σ.
- Substitutability, $\mathcal{S}(V)$, is independent of V and equal to a constant, $\sigma > 1$.
- LV, $\mathcal{L}(V)$, is independent of V, and equal to a constant, $\mathcal{L}(V) = \mathcal{L} = 1/(\sigma - 1)$, inversely related to σ.

Digression: Generalized CES with Gross Substitutes a la Benassy (1996).

$$X(\mathbf{x}) = Z(V)\left[\int_{\Omega} x_{\omega}^{1-\frac{1}{\sigma}} d\omega\right]^{\frac{\sigma}{\sigma-1}} \Leftrightarrow P(\mathbf{p}) = \frac{1}{Z(V)}\left[\int_{\Omega} p_{\omega}^{1-\sigma} d\omega\right]^{\frac{1}{1-\sigma}},$$

Note: $Z(V)$ allows variety to have direct externalities to TFP (or affinity)

	Under Generalized CES
Budget Share	$s_{\omega} = \left(\frac{p_{\omega}}{Z(V)P(\mathbf{p})}\right)^{1-\sigma} = \left(\frac{Z(V)x_{\omega}}{X(\mathbf{x})}\right)^{1-1/\sigma}$
Price Elasticity	$\zeta_{\omega} = \sigma > 1$
Substitutability	$\mathcal{S}(V) = \sigma > 1$
Love-for-variety	$\mathcal{L}(V) = \frac{1}{\sigma - 1} + \frac{d \ln Z(V)}{d \ln V}.$

- PE, ζ_{ω}, and Substitutability, $\mathcal{S}(V)$, are not affected by $d \ln Z(V)/d \ln V$, "the Benassy residual", which "accounts for" the gap btw CES-implied LV (say, from the markup) & revealed LV (say, from productivity growth).
- Benassy (1996) set $d \ln Z(V)/d \ln V = \nu - 1/(\sigma - 1)$, so that $\mathcal{L}(V) = \nu$ is a separate parameter.

Even if you believe in the direct externalities behind the Benassy residual, your estimate of its magnitude depends on the CES assumption, which nobody believes.

In all the non-CES considered below, we could also let TFP vary directly with V, which would add the term, $d \ln Z(V)/d \ln V$, to the expression for $\mathcal{L}(V)$, without affecting the expression for $\mathcal{S}(V)$.

Comparing Substitutability and Love-for-Variety Measures

In general, the relation btw $\mathcal{S}(V)$, & $\mathcal{L}(V)$ can be complex. For example, $\mathcal{S}(V)$ and $\mathcal{L}(V)$ could be positively related. To see why, let's compare their definitions side-by-side.

	$\mathcal{S}(V)$	$\mathcal{L}(V)$
In terms of $P(\mathbf{p})$	$1-\frac{\partial}{\partial \ln p_\omega}\ln\left[\frac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega}\right]\Big\|_{\mathbf{p}=p\mathbf{1}_\Omega^{-1},\ p=const.}$	$-\frac{d\ln P(\mathbf{p})}{d\ln V}\Big\|_{\mathbf{p}=p\mathbf{1}_\Omega^{-1},\ p=const.}$
In terms of $X(\mathbf{x})$	$1-\frac{\partial}{\partial \ln x_\omega}\ln\left[\frac{\partial \ln X(\mathbf{x})}{\partial \ln x_\omega}\right]\Big\|_{\mathbf{x}=x\mathbf{1}_\Omega,\ x=const.}$	$\frac{d\ln X(\mathbf{x})}{d\ln V}\Big\|_{\mathbf{x}=x\mathbf{1}_\Omega, xV=const.}$

$\mathcal{L}(V)$ captures the curvature of **the utility (production) function.**

$\mathcal{S}(V)$ captures the curvature of the budget share function, which is related to ***marginal*** **utility (production) function.**

Moreover, the relation btw $\zeta(p_\omega;\mathbf{p}) = \zeta^*(x_\omega;\mathbf{x})$ and $\mathcal{S}(V)$ or $\mathcal{L}(V)$ can be complex. For example, whether the 2nd Law holds or not says little about the derivatives of $\mathcal{S}(V)$ and $\mathcal{L}(V)$.

To make further progress, we turn to 5 classes of demand systems, that are pairwise disjoint with the exception of CES.

Two Classes of Geometric Means of CES

Two Versions of GM-CES

$G(\cdot)$: the cdf of $\sigma \in (1, \infty)$, and $\mathbb{E}_G[f(\sigma)]$: the expected value of $f(\sigma)$.

Weighted Geometric Means of Symmetric CES (GM-CES) Unit Cost Function

$$\ln P(\mathbf{p}) \equiv \int_1^\infty \ln P(\mathbf{p};\sigma)\, dG(\sigma) \equiv \mathbb{E}_G[\ln P(\mathbf{p};\sigma)] \qquad \text{where} \qquad [P(\mathbf{p};\sigma)]^{1-\sigma} \equiv \int_\Omega {p_\omega}^{1-\sigma}\, d\omega$$

Weighted Geometric Means of Symmetric CES (GM-CES) Production Function

$$\ln X(\mathbf{x}) \equiv \int_1^\infty \ln X(\mathbf{x};\sigma)\, dG(\sigma) \equiv \mathbb{E}_G[\ln X(\mathbf{x};\sigma)] \qquad \text{where} \qquad [X(\mathbf{x};\sigma)]^{1-\frac{1}{\sigma}} \equiv \int_\Omega {x_\omega}^{1-\frac{1}{\sigma}}\, d\omega$$

Clearly, both satisfy linear homogeneity, strict quasi-concavity, and symmetry.

	GM-CES Unit Cost Function	**GM-CES Production Function**
Budget Share	$s(p_\omega;\mathbf{p}) = \mathbb{E}_G\left[\left(\frac{p_\omega}{P(\mathbf{p};\sigma)}\right)^{1-\sigma}\right]$	$s^*(x_\omega;\mathbf{x}) = \mathbb{E}_G\left[\left(\frac{x_\omega}{X(\mathbf{x};\sigma)}\right)^{1-1/\sigma}\right]$
Price Elasticity	$\zeta(p_\omega;\mathbf{p}) = \frac{\mathbb{E}_G[\sigma {p_\omega}^{-\sigma}/[P(\mathbf{p};\sigma)]^{1-\sigma}]}{\mathbb{E}_G[{p_\omega}^{-\sigma}/[P(\mathbf{p};\sigma)]^{1-\sigma}]}$	$\zeta^*(x_\omega;\mathbf{x}) = \frac{\mathbb{E}_G\left[(x_\omega)^{-1/\sigma}/[X(\mathbf{x};\sigma)]^{1-1/\sigma}\right]}{\mathbb{E}_G\left[(x_\omega)^{-1/\sigma}/\sigma[X(\mathbf{x};\sigma)]^{1-1/\sigma}\right]}$
Substitutability	$\mathcal{S}(V) = \mathbb{E}_G[\sigma]$	$\mathcal{S}(V) = \frac{1}{\mathbb{E}_G[1/\sigma]}$
Love-for-Variety	$\mathcal{L}(V) = \mathbb{E}_G\left[\frac{1}{\sigma-1}\right] \geq \frac{1}{\mathcal{S}(V)-1}$	$\mathcal{L}(V) = \mathbb{E}_G\left[\frac{1}{\sigma-1}\right] \geq \frac{1}{\mathcal{S}(V)-1}$

Note: GM-CES are *not* nested CES.

Theorem 1 (GM-CES): Under the two classes of GM-CES demand systems,
1-i): $\mathcal{S}(V)$ and $\mathcal{L}(V)$ are both constant, given by:

$$\mathcal{S}(V) = \mathbb{E}_G[\sigma] > 1;\;\; \mathcal{L}(V) = \mathbb{E}_G\left[\frac{1}{\sigma - 1}\right] > 0 \qquad \text{under GM-CES unit cost function:}$$

$$\mathcal{S}(V) = \frac{1}{\mathbb{E}_G[1/\sigma]} > 1;\;\; \mathcal{L}(V) = \mathbb{E}_G\left[\frac{1}{\sigma - 1}\right] > 0 \qquad \text{under GM-CES production function.}$$

1-ii): $\mathcal{L}(V)$ can be arbitrarily large, without any upper bound, while its lower bound is given by:

$$\mathcal{L}(V) \geq \frac{1}{\mathcal{S}(V) - 1} > 0.$$

where the equality holds if and only if $G(\cdot)$ is degenerate, i.e., only under CES.

Notes:

- For a non-degenerate $G(\cdot)$, Jensen's inequality implies:

$$\mathcal{L}(V) - \frac{1}{\mathcal{S}(V) - 1} > 0; \qquad \mathbb{E}_G[\sigma] > \frac{1}{\mathbb{E}_G[1/\sigma]}$$

  - The 1st inequality may be interpreted as offering a microfoundation for the Benassy residual.
  - The CES formula for LV underestimates LV under GM-CES or thus overestimates the Benassy residuals and/or quality improvement term.
  - The 2^{nd} inequality implies that CES is the only intersection of the two classes of GM-CMS.

- There exist any number of families of cdf's, G, such that $\mathcal{S}(V)$ and $\mathcal{L}(V)$ are positively related within each family.

Three Classes: H.S.A., HDIA, and HIIA

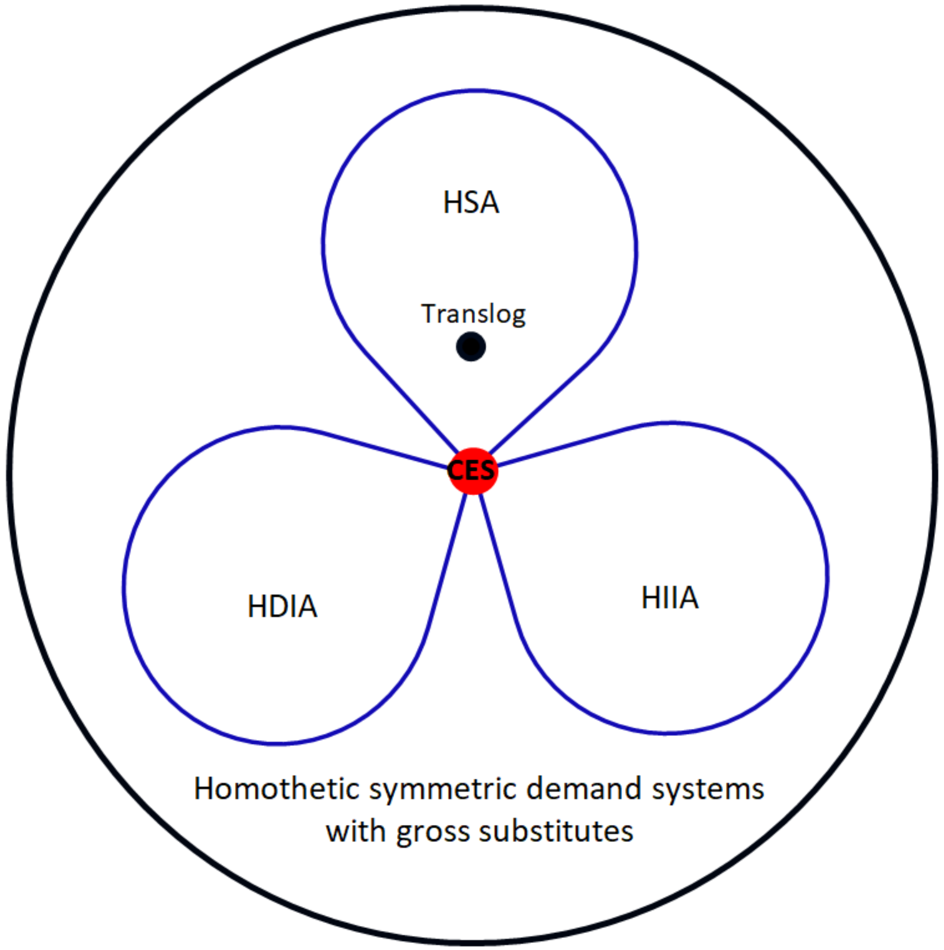
Intuitively, one might think that, as variety increases,

- PE of demand for each good become larger.
- Different goods become more substitutable.
- LV becomes smaller.

Homotheticity is too general to capture this intuition!!
It is NOT restrictive enough.

To capture this intuition, we turn to

✓ **Homothetic Single Aggregator (H.S.A.)**
✓ **Homothetic Direct Implicit Additivity (HDIA)**
✓ **Homothetic Indirect Implicit Additivity (HIIA)**



3 Classes of Symmetric Homothetic Demand Systems (with gross Substitutes & Inessentiality)

$\mathcal{M}[\cdot]$ is a monotone transformation.

Homothetic Direct Implicit Additivity (HDIA): $\phi(\cdot)$	$\mathcal{M}\left[\int_{\Omega} \phi\left(\frac{Zx_{\omega}}{X(\mathbf{x})}\right) d\omega\right] \equiv \mathcal{M}\left[\int_{\Omega} \phi\left(\frac{x_{\omega}}{\hat{X}(\mathbf{x})}\right) d\omega\right] \equiv 1.$

$\phi(\cdot): \mathbb{R}_{+} \to \mathbb{R}_{+}$, thus $\hat{X}(\mathbf{x})$, is independent of $Z > 0$, TFP.
$\phi(0) = 0; \phi(\infty) = \infty; \phi'(\infty) = 0; \phi'(y) > 0 > \phi''(y), 0 < -y\phi''(y)/\phi'(y) < 1$, for $\forall y \in (0, \infty)$.
CES with $\phi(y) = (y)^{1-1/\sigma}, \sigma > 1$. The choke price exists if $\phi'(0) < \infty$.

Homothetic Indirect Implicit Additivity (HIIA): $\theta(\cdot)$	$\mathcal{M}\left[\int_{\Omega} \theta\left(\frac{p_{\omega}}{ZP(\mathbf{p})}\right) d\omega\right] \equiv \mathcal{M}\left[\int_{\Omega} \theta\left(\frac{p_{\omega}}{\hat{P}(\mathbf{p})}\right) d\omega\right] \equiv 1.$

$\theta(\cdot): \mathbb{R}_{++} \to \mathbb{R}_{+}$, thus $\hat{P}(\mathbf{p})$, is independent of $Z > 0$ is TFP.
$\theta(z) > 0, \theta'(z) < 0 < \theta''(z), -z\theta''(z)/\theta'(z) > 1$ for $0 < z < \bar{z} \leq \infty$, $\theta(0) = \infty; \theta(z) = \theta'(z) = 0$ for $z \geq \bar{z}$.
CES with $\theta(z) = (z)^{1-\sigma}, \sigma > 1$. The choke price exists if $\bar{z} < \infty$.

Homothetic Single Aggregator (H.S.A.): $s(\cdot)$	$s_{\omega} = \frac{\partial \ln P(\mathbf{p})}{\partial \ln p_{\omega}} = s\left(\frac{p_{\omega}}{A(\mathbf{p})}\right)$ with $\int_{\Omega} s\left(\frac{p_{\omega}}{A(\mathbf{p})}\right) d\omega \equiv 1.$

$s(\cdot): \mathbb{R}_{+} \to \mathbb{R}_{+}$, thus $A(\mathbf{p})$, is independent of $Z > 0$, TFP.
$s(z) > 0 > s'(z)$ for $0 < z < \bar{z} \leq \infty$; $s(z) = 0$ for $z \geq \bar{z}$. $s(0) = \infty$ to be well-defined for any arbitrarily small $V > 0$.
CES with $s(z) = \gamma z^{1-\sigma}, \sigma > 1$. The choke price exists if $\bar{z} < \infty$.
$Z > 0$ shows up when integrating the budget share $s(p_{\omega}/A(\mathbf{p}))$ to obtain $P(\mathbf{p})$ or $X(\mathbf{x})$.

Key Properties of the Three Classes

	Budget Shares: $s_\omega \equiv \frac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} = s(p_\omega; \mathbf{p})$		**Price Elasticity:** $\zeta_\omega \equiv -\frac{\partial \ln x_\omega}{\partial \ln p_\omega} = \zeta(p_\omega; \mathbf{p})$
CES	$s_\omega = \left(\frac{p_\omega}{ZP(\mathbf{p})}\right)^{1-\sigma}$		σ
H.S.A. $s(\cdot)$	$s_\omega = s\left(\frac{p_\omega}{A(\mathbf{p})}\right)$	$\frac{P(\mathbf{p})}{A(\mathbf{p})} \neq c$, unless CES	$\zeta^S\left(\frac{p_\omega}{A(\mathbf{p})}\right)$; $\zeta^S(z) \equiv 1 - \frac{zs'(z)}{s(z)} > 1$
HDIA $\phi(\cdot)$	$s_\omega = \frac{p_\omega}{P(\mathbf{p})}(\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right)$	$\frac{P(\mathbf{p})}{B(\mathbf{p})} \neq c$, unless CES	$\zeta^D\left((\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right)\right)$; $\zeta^D(y) \equiv -\frac{\phi'(y)}{y\phi''(y)} > 1$
HIIA $\theta(\cdot)$	$s_\omega = \frac{p_\omega}{C(\mathbf{p})}\theta'\left(\frac{p_\omega}{P(\mathbf{p})}\right)$	$\frac{P(\mathbf{p})}{C(\mathbf{p})} \neq c$, unless CES	$\zeta^I\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right)$; $\zeta^I(z) \equiv -\frac{z\theta''(z)}{\theta'(z)} > 1.$

$A(\mathbf{p}), B(\mathbf{p}), C(\mathbf{p})$: each defined implicitly by the adding-up constraint, $\int_\Omega s_\omega d\omega \equiv 1$. Clearly, they are all linear homogenous.

We focus on these three classes for two reasons.

- They are pairwise disjoint with the sole exception of CES.
- PE $= \zeta_\omega \equiv \zeta\left(\frac{p_\omega}{\mathcal{A}(\mathbf{p})}\right)$, where $\mathcal{A}(\mathbf{p})$ is linear homogenous, a sufficient statistic, capturing all the cross-product effects.

Key Properties of the Three Classes, Continued.

	Price Elasticity: $\zeta(p_\omega;\mathbf{p})$	Substitutability: $\mathcal{S}(V)$	Love-for-Variety: $\mathcal{L}(V)$
H.S.A.	$\zeta_\omega = \zeta^S\left(\frac{p_\omega}{A(\mathbf{p})}\right)$	$\zeta^S\left(s^{-1}\left(\frac{1}{V}\right)\right)$	$\Phi\left(s^{-1}\left(\frac{1}{V}\right)\right) = \frac{1}{\mathcal{E}_H\left(s^{-1}(1/V)\right)},$

where $\zeta^S(z) \equiv -\frac{zH''(z)}{H'(z)} > 1$ and $\frac{1}{\Phi(z)} = \mathcal{E}_H(z) \equiv -\frac{zH'(z)}{H(z)} > 0$, with $H(z) \equiv \int_z^{\bar{z}} \frac{s(\xi)}{\xi} \mathrm{d}\xi > 0$.

HDIA	$\zeta_\omega = \zeta^D\left((\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right)\right)$	$\zeta^D\left(\phi^{-1}\left(\frac{1}{V}\right)\right)$	$\frac{1}{\mathcal{E}_\phi(\phi^{-1}(1/V))} - 1$

where $\zeta^D(y) \equiv -\frac{\phi'(y)}{y\phi''(y)} > 1$ and $0 < \mathcal{E}_\phi(y) \equiv \frac{y\phi'(y)}{\phi(y)} < 1$.

HIIA	$\zeta_\omega = \zeta^I\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right)$	$\zeta^I\left(\theta^{-1}\left(\frac{1}{V}\right)\right)$	$\frac{1}{\mathcal{E}_\theta\left(\theta^{-1}(1/V)\right)}$

where $\zeta^I(z) \equiv -\frac{z\theta''(z)}{\theta'(z)} > 1$ and $\mathcal{E}_\theta(z) \equiv -\frac{z\theta'(z)}{\theta(z)} > 0$.

Note: In all three classes,

- $\mathcal{L}(V)$ depends on the curvature of a function of a single variable, $H(\cdot), \phi(\cdot), \theta(\cdot)$
- $\mathcal{S}(V)$ depends on the curvature of its derivative. $H'(\cdot), \phi'(\cdot), \theta'(\cdot)$.

Theorem 2: Under H.S.A., HDIA, & HIIA,

2-i) $\mathcal{S}'(V) > 0$ iff the 2nd law holds.

2-ii) $\mathcal{S}'(V) \gtreqless 0$ for all $V \in (V_0, \infty) \Longrightarrow \mathcal{L}'(V) \lesseqgtr 0$ for all $V \in (V_0, \infty)$.

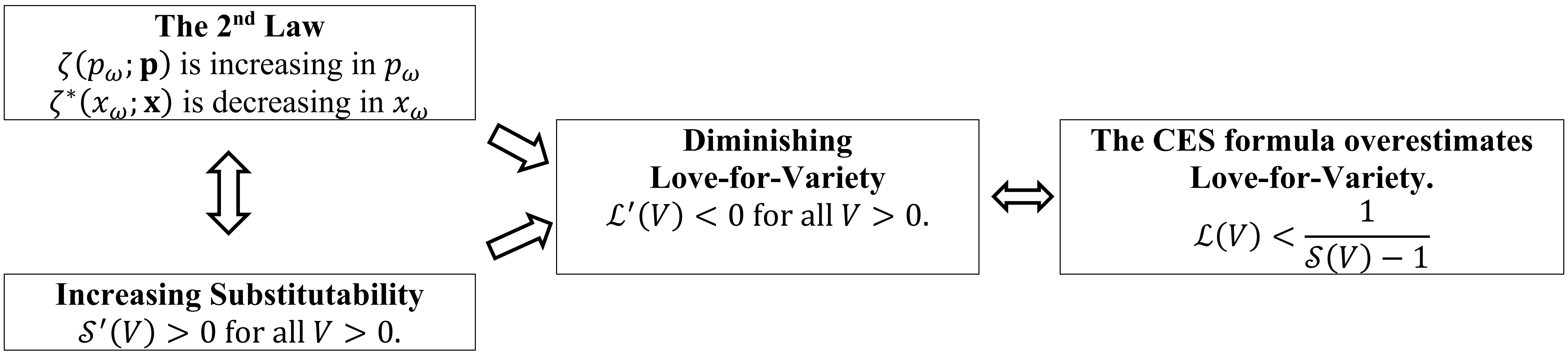
The converse is not true in general. However,

2-iii) $\mathcal{L}'(V) = 0$ for all $V \in (V_0, \infty) \Leftrightarrow \mathcal{S}'(V) = 0$ for all $V \in (V_0, \infty)$.

In particular, $\mathcal{L}'(V) = 0$ for all $V > 0 \Leftrightarrow \mathcal{S}'(V) = 0$ for all $V > 0 \Leftrightarrow$ CES.

Theorem 3: Under H.S.A., HDIA, & HIIA,

$$\mathcal{L}'(V) \lesseqgtr 0 \Leftrightarrow \mathcal{L}(V) \lesseqgtr \frac{1}{\mathcal{S}(V)-1} > 0$$



Theorem 4: Under H.S.A., HDIA, & HIIA, $\lim_{V\to\infty} \mathcal{L}(V) = \lim_{V\to\infty} \frac{1}{\mathcal{S}(V)-1}$. In particular, $\lim_{V\to\infty} \mathcal{S}(V) = \infty \Leftrightarrow \lim_{V\to\infty} \mathcal{L}(V) = 0$.

An Application to an Armington Model of Trade

An Armington Model of Competitive Trade:

Two Countries: Home & Foreign* differ only in labor supply L & L^*(with the wage rates, w & w^*) and goods they produce, Ω & Ω^*; $\Omega \cap \Omega^* = \emptyset$, with $V \equiv |\Omega|$ & $V^* \equiv |\Omega^*|$.

Technology: One unit of Home (Foreign) labor produces one unit of each Home (Foreign) good.

No Trade Cost: In both countries, the unit prices of goods are $p_\omega = w$ $(\omega \in \Omega)$ and $p^*_\omega = w^*$ $(\omega \in \Omega^*)$.

Symmetric Homothetic Demand: Asymmetry of countries are due to the variety of goods they can produce.
D &M: Home demand for each Home & Foreign good; D^* & M^*: Foreign demand for each Foreign & Home good.

	Home	**Foreign**
Resource Constraint:	$V(D + M^*) = L$	$V^*(M + D^*) = L^*$
Budget Constraint:	$wVD + w^*V^*M = wL$	$wVM^* + w^*V^*D^* = w^*L^*.$
Trade-GDP Ratio:	$\dfrac{w^*V^*M}{wL} = \dfrac{wVM^*}{wL} = \dfrac{VM^*}{L}$	$\dfrac{wVM^*}{w^*L^*} = \dfrac{w^*V^*M}{w^*L^*} = \dfrac{V^*M}{L^*}$

Balanced Trade	$wVM^* = w^*V^*M.$
Relative Supply = Relative Demand:	$\dfrac{L/V}{L^*/V^*} = RS = RD = \dfrac{D + M^*}{M + D^*} = \dfrac{D}{M} = \dfrac{M^*}{D^*} = g\left(\dfrac{w}{w^*}; V; V^*\right) \lesseqgtr 1 \Leftrightarrow \dfrac{w}{w^*} \gtreqless 1.$

Consider the case where the two countries differ proportionally in size with f being the Home's share.

$$\frac{L}{L^*} = \frac{V}{V^*}$$

Then, from the RS = RD condition,

$$\frac{L/V}{L^*/V^*} = 1 \Leftrightarrow \frac{w}{w^*} = 1 \Leftrightarrow \frac{D}{M} = \frac{M^*}{D^*} = 1.$$

Balanced Trade condition becomes $VM^* = V^*M$.

$$\frac{L}{L^*} = \frac{V}{V^*} = \frac{M}{M^*} = \frac{D}{D^*} \Leftrightarrow \frac{V}{L} = \frac{V^*}{L^*}; \frac{D}{L} = \frac{D^*}{L^*} = \frac{M}{L} = \frac{M^*}{L^*}.$$

Per capita term, Home and Foreign become identical.

	Home	**Foreign**
Domestic Expenditure Share	$\lambda = \frac{V}{V + V^*}$	$\lambda^* = \frac{V^*}{V + V^*}$
Trade-GDP Ratio	$1 - \lambda = \frac{V^*}{V + V^*}$	$1 - \lambda^* = \frac{V}{V + V^*}$

Gains from Trade: Equivalent to $V \rightarrow V + V^* = V/\lambda$ for Home and to $V^* \rightarrow V + V^* = V^*/\lambda^*$ for Foreign.

	Home	**Foreign**
Gains from Trade	$GT \equiv \frac{P\left(\mathbf{1}_{\Omega}^{-1}\right)}{P(\mathbf{1}_{\Omega\cup\Omega^*}^{-1})} = \exp\left[\int_{V}^{V/\lambda} \mathcal{L}(v)\frac{dv}{v}\right]$	$GT^* \equiv \frac{P\left(\mathbf{1}_{\Omega^*}^{-1}\right)}{P(\mathbf{1}_{\Omega\cup\Omega^*}^{-1})} = \exp\left[\int_{V^*}^{V^*/\lambda^*} \mathcal{L}(v)\frac{dv}{v}\right]$

The effect of Home openness, $\lambda = V/(V + V^*) \downarrow$, on Home GT may depend on whether it is due to $V \downarrow$ or $V^* \uparrow$.

General Implications:

Theorem 5 (The Effects of Country Sizes, V and V^* on Gains from Trade):

5-i) GT is larger for the smaller country than for the larger country.

$$GT \gtreqless GT^* \Leftrightarrow V \lesseqgtr V^* \Leftrightarrow \lambda \lesseqgtr \lambda^*$$

5-ii) If the two countries are proportionately larger, GT are diminishing for both countries under diminishing LV.

$$\left.\frac{\partial \ln(GT)}{\partial \ln V}\right|_{\lambda=const.} = \mathcal{L}(V/\lambda) - \mathcal{L}(V) < 0; \qquad \left.\frac{\partial \ln(GT^*)}{\partial \ln V^*}\right|_{\lambda^*=const.} = \mathcal{L}(V^*/\lambda^*) - \mathcal{L}(V^*) < 0;$$

5-iii) For any given V, GT is increasing in V^* (thus decreasing in λ),

$$\left.\frac{\partial \ln(GT)}{\partial \ln V^*}\right|_{V=const.} = (1-\lambda)\mathcal{L}(V/\lambda) > 0$$

with the range,

$$0 < \ln(GT) < \int_V^{\infty} \mathcal{L}(v)\frac{dv}{v}.$$

The upper bound is infinite if $\mathcal{L}(\infty) > 0$. It may be finite if $\mathcal{L}(\infty) = 0$. If finite, the upper bound is decreasing in V.

5-iv) For any given V^*, GT may be nonmonotone in V (thus in λ) in general. Under non-increasing LV,

$$\left.\frac{\partial \ln(GT)}{\partial \ln V}\right|_{V^*=const.} = \lambda\mathcal{L}(V/\lambda) - \mathcal{L}(V) < 0$$

hence GT is decreasing in V (thus in λ), with the range

$$0 < \ln(GT) < \int_0^{V^*} \mathcal{L}(v)\frac{dv}{v}.$$

The upper bound is finite if $\mathcal{L}(0) < \infty$. It may be infinite if $\mathcal{L}(0) = \infty$. If finite, the upper bound is increasing in V^*.

Gains from Trade under CES

Substitutability: $\mathcal{S}(V)$	Love-for-Variety: $\mathcal{L}(V)$	Home Gains from Trade
$\mathcal{S}^{CES} = \sigma > 1$	$\mathcal{L}^{CES} = \frac{1}{\sigma - 1}$	$\ln(GT) = \mathcal{L}^{CES} \ln\left(\frac{1}{\lambda}\right) = \frac{1}{\mathcal{S}^{CES} - 1} \ln\left(\frac{1}{\lambda}\right)$

- $\mathcal{S}(V)$ and $\mathcal{L}(V)$ are constant under CES.
- GT satisfies the familiar ACR formula.
- Decreasing in λ (thus increasing in the openness, $1 - \lambda$).
- GT goes to infinity as $\lambda \to 0$, or $V/V^* \to 0$. Once λ is controlled for, V and V^* play no role.

Gains from Trade under GM-CES

Substitutability: $\mathcal{S}(V)$	Love-for-Variety: $\mathcal{L}(V)$	Home Gains from Trade
$\mathcal{S}^{GMCES} = \mathbb{E}_G[\sigma]$ for GM-CES unit cost fn. $\mathcal{S}^{GMCES} = \left[\mathbb{E}_G[1/\sigma]\right]^{-1}$ for GM-CES production fn.	$\mathcal{L}^{GMCES} = \mathbb{E}_G\left[\frac{1}{\sigma - 1}\right]$	$\ln(GT) = \mathcal{L}^{GMCES} \ln\left(\frac{1}{\lambda}\right) \geq \frac{1}{\mathcal{S}^{GMCES} - 1} \ln\left(\frac{1}{\lambda}\right)$

- $\mathcal{S}(V)$ and $\mathcal{L}(V)$ are also constant under GM-CES.
- GT satisfies the familiar ACR formula, with $\mathcal{L}^{GMCES}$ but not with $\mathcal{S}^{GMCES}$.

For the ACR formula, what is crucial is that LV is constant.

- GT is decreasing in λ (thus increasing in the openness, $1 - \lambda$).
- GT goes to infinity as $\lambda \to 0$, or $V/V^* \to 0$. Once λ is controlled for, V and V^* play no role.
- For any level of λ, GT under GM-CES can be arbitrarily large, with GT under CES being the lower bound.

If one views $\mathcal{S}^{GMCES}$ being constant as the evidence for CES, one would underestimate GT under GM-CES.

Gains from Trade under H.S.A.

Substitutability: $\mathcal{S}(V)$	Love-for-Variety: $\mathcal{L}(V)$	Home Gains from Trade
$\zeta^S\left(s^{-1}\left(\frac{1}{V}\right)\right)$	$\Phi\left(s^{-1}\left(\frac{1}{V}\right)\right)$	$GT = \frac{s^{-1}(\lambda/V)}{s^{-1}(1/V)}\frac{\exp[\Phi(s^{-1}(\lambda/V))]}{\exp[\Phi(s^{-1}(1/V))]}$

- For a given V, $V^* \uparrow$ (thus $\lambda \downarrow$) increases Home GT, with the upper bound

$$GT < \frac{\bar{z}}{s^{-1}(1/V)}\frac{\exp[\Phi(\bar{z})]}{\exp[\Phi(s^{-1}(1/V))]} < \infty \iff \bar{z} < \infty.$$

If finite, the upper bound is decreasing in V. CES & GM-CES overestimate gains from trade with a large country.

- For a given V^*, $V \downarrow$ (thus $\lambda \downarrow$) increases Home GT, under non-increasing LV. The upper bound is infinite.

$$GT < \frac{s^{-1}(1/V^*)}{s^{-1}(\infty)}\frac{\exp[\Phi(1/V^*)]}{\exp[\Phi(s^{-1}(\infty))]} = \infty.$$

Parametric Examples of H.S.A. All feature the 2nd law, Increasing Substitutability, Diminishing LV, the choke price.

	$\mathcal{S}(V)$	Love-for-Variety: $\mathcal{L}(V)$	Home Gains from Trade
Translog	$1+\gamma V$	$\frac{1/2}{\gamma V}$	$\ln(GT) = \frac{1-\lambda}{2\gamma V}$
Generalized Translog	$1+(\sigma-1)(\gamma V)^{1/\eta}$	$\frac{\eta/(1+\eta)}{(\sigma-1)(\gamma V)^{1/\eta}}$	$\ln(GT) = \frac{(\gamma V)^{-1/\eta}}{\sigma-1}\frac{\eta}{1+\eta}\frac{1-(\lambda)^{1/\eta}}{1/\eta}$
CoPaTh	$\sigma(\gamma V)^{\frac{1-\rho}{\rho}} > 1$	$\sum_{n=0}^{\infty}\frac{\rho}{1+(1-\rho)n}\left[\frac{1}{\sigma(\gamma V)^{\frac{1-\rho}{\rho}}}\right]^n$	$\ln(GT) = -\sum_{n=0}^{\infty}\frac{\rho}{1+(1-\rho)n}\frac{1-(\lambda)^{\frac{1-\rho}{\rho}(n+1)}}{\left[\sigma(\gamma V)^{\frac{1-\rho}{\rho}}\right]^{(n+1)}} + \ln\left[1+\frac{1-(\lambda)^{\frac{1-\rho}{\rho}}}{\sigma(\gamma V)^{\frac{1-\rho}{\rho}}-1}\right]^{\frac{\rho}{1-\rho}}$

Generalized Translog $(0 < \eta < \infty)$**:** The case of $\eta = 1$ is isomorphic to Translog. CES is the limit case, $\eta \to \infty$.
CoPaTh $(0 < \rho < 1)$**:** CES is the limit case, $\rho \to 1$.

Gains from Trade under HDIA

Substitutability: $\mathcal{S}(V)$	Love-for-Variety: $\mathcal{L}(V)$	Gains from Trade
$\zeta^D\left(\phi^{-1}\left(\frac{1}{V}\right)\right)$	$\frac{1}{\mathcal{E}_\phi(\phi^{-1}(1/V))} - 1$	$GT = \frac{(\lambda/V)}{\phi^{-1}(\lambda/V)}\frac{\phi^{-1}(1/V)}{(1/V)}$

- For a given V, $V^* \uparrow$ (thus $\lambda \downarrow$) increases Home GT, with the upper bound

$$GT < \phi'(0)\frac{\phi^{-1}(1/V)}{(1/V)} < \infty \Leftrightarrow \phi'(0) < \infty.$$

If finite. the upper bound is decreasing in V. CES and GM-CES overestimate gains from trade with a large country.

- For a given V^*, $V \downarrow$ (thus $\lambda \downarrow$) increases Home GT, under non-increasing LV. The upper bound is infinite.

$$GT < \frac{(1/V^*)}{\phi^{-1}(1/V^*)}\frac{1}{\phi'(\infty)} = \infty.$$

Gains from Trade under HIIA

Substitutability: $\mathcal{S}(V)$	Love-for-Variety: $\mathcal{L}(V)$	Gains from Trade
$\mathcal{S}^I(V) = \zeta^I\left(\theta^{-1}\left(\frac{1}{V}\right)\right)$	$\mathcal{L}^I(V) = \frac{1}{\mathcal{E}_\theta\left(\theta^{-1}(1/V)\right)}$	$GT = \frac{\theta^{-1}(\lambda/V)}{\theta^{-1}(1/V)}$

- For a given V, $V^* \uparrow$ (thus $\lambda \downarrow$) increases Home GT, with the upper bound

$$GT < \frac{\bar{z}}{\theta^{-1}(1/V)} < \infty \Leftrightarrow \bar{z} < \infty.$$

If finite. the upper bound is decreasing in V. CES and GM-CES overestimate gains from trade with a large country.

- For a given V^*, $V \downarrow$ (thus $\lambda \downarrow$) increases Home GT, under non-increasing LV. The upper bound is infinite.

$$GT < \frac{\theta^{-1}(1/V^*)}{\theta^{-1}(\infty)} = \infty.$$

Concluding Remarks

What We Did in This Paper

- Investigated how LV depends on the underlying demand structure outside of CES.

- Defined **Substitutability** & **Love-for-Variety** measures, both depend only on V under homotheticity & symmetry

- **GM-CES:** Both measures are constant like CES, but the CES formula would underestimate LV under GM-CES (and overestimate the Benassy residuals and/or quality improvement term).

- **3 classes (H.S.A., HDIA, HIIA):**
    - 2nd Law ⟺ Increasing Substitutability ⟹ Diminishing LV⟹ The CES formula would overestimate LV (and underestimate the Benassy residuals and/or quality improvement term)
    - LV goes asymptotically to zero, as V goes to infinity, if the choke price exists

- We illustrated some implications on gains from trade (GT) in a simple Armington model of trade.
    - GM-CES: Though ACR formula holds, CES underestimate GT, controlling for the openness.
    - H.S.A. HDIA and HIIA with the choke price. GT is increasing in the size of the trading partner, but it is bounded, unlike CES. CES may overestimate gains from trade with a large country.

Some Extensions

- **Geometric Means of HSA/HDIA/HIIA with Increasing Substitutability**

- **Nonhomothetic Preferences**

Other Applications

- **Implications on Gravity Law:** Armington models with finite trade costs. This would require parametric restrictions.

- **Static Monopolistic Competition:** Under GM-CES, insufficient entry. Under all 3 classes, the 2nd Law $\Leftrightarrow$ Increasing Substitutability $\Leftrightarrow$ Procompetitive Entry $\Longrightarrow$ Excessive Entry, as shown in Matsuyama-Ushchev (2020), which we need to revise.

- **Romer-type Endogenous Growth with Expanding Variety/Knowledge Spillover**
  - Under CES and GM-CES, too little R&D in equilibrium.
  - Under the 3 classes with the 2nd law, R&D can be too much in equilibrium, as in a vertical innovation model.